Pair Creation of Black Holes in Anti-de Sitter Space Background

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Abstract

In the absence of a general no-boundary proposal for open creation, the complex constrained instanton is used as the seed for the open pair creations of black holes in the Kerr-Newman-anti-de Sitter family. The relative creation probability of the chargeless and nonrotating black hole pair is the exponential of the negative of the entropy, and that of the charged and (or) rotating black hole pair is the exponential of the negative of one quarter of the sum of the outer and inner black hole horizon areas.

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In the No-Boundary Universe, the wave function of a closed universe is defined as a path integral over all compact 4-metrics with matter fields [1]. The dominant contribution to the path integral is from the stationary action solution. At the WKB level, the wave function can be written as

$$\Psi \approx e^{-I}, \tag{1}$$

where $I = I_r + iI_i$ is the complex action of the solution.

The Euclidean action is

$$I = -\frac{1}{16\pi}\int_M (R - 2\Lambda + L_m) - \frac{1}{8\pi}\oint_{\partial M} K, \tag{2}$$

where R is the scalar curvature of the spacetime M, Λ is the cosmological constant, K is the trace of the second form of the boundary ∂M, and L_m is the Lagrangian of the matter content.

The imaginary part I_i and real part I_r of the action represent the Lorentzian and Euclidean evolutions in real time and imaginary time, respectively. When their orbits are intertwined, they are mutually perpendicular in the configuration space with the supermetric. The probability of a Lorentzian orbit remains constant during the evolution. One can identify the probability, not only as the probability of the universe created, but also as the probabilities for other Lorentzian universes obtained through an analytic continuation from it [2].

An instanton is defined as a stationary action orbit and satisfies the Einstein equation everywhere. It is the seed for the creation of the universe. However, very few regular instantons exist. The framework of the No-Boundary Universe is much wider than that of the instanton theory. Therefore, in order not to exclude many interesting phenomena from the study, one has to appeal to the concept of constrained instantons [3]. Constrained instantons are the orbits with an action which is stationary under some restriction. The restriction can be imposed on a spacelike 3-surface of the created Lorentzian universe. The restriction is that the 3-metric and matter content are given at the 3-surface. The relative creation probability from the instanton is the exponential of the negative of the real part of the instanton action.

The usual prescription for finding a constrained instanton is to obtain a complex solution to the Einstein equation and other field equations in the complex domain of spacetime coordinates. If there

is no singularity in a compact section of the solution, then the compact section of the solution is considered as an instanton. If there exist singularities in the section, then the action of the section is not stationary. The action may only be stationary with respect to the variations under some restrictions mentioned above. If this is the case, then the section is a constrained gravitational instanton. To find the constrained instanton, one has to closely investigate the singularities. The stationary action condition is crucial to the validation of the WKB approximation. We are going to work at the WKB level for the problem of quantum creation of a black hole pair.

A main unresolved problem in quantum cosmology is to generalize the no-boundary proposal for an open universe. While a general prescription is not available, one can still use analytic continuation to obtain the WKB approximation to the wave function for open universes with some kind of symmetry. The S^4 space model with $O(5)$ symmetry [4] and the $FLRW$ space model with $O(4)$ symmetry [2] have been discussed.

In this paper, we try to reduce the symmetry further, and study the problem of quantum pair creation of black holes in the Kerr-Newman-anti-de Sitter family. Let us discuss the most general case, i.e. the quantum creation of the Kerr-Newman-anti-de Sitter black hole pair. The Lorentzian metric of the black hole spacetime is [5]

$$ds^2 = \rho^2(\Delta_r^{-1}dr^2+\Delta_\theta^{-1}d\theta^2)+\rho^{-2}\Xi^{-2}\Delta_\theta \sin^2\theta(adt-(r^2+a^2)d\phi)^2-\rho^{-2}\Xi^{-2}\Delta_r(dt-a\sin^2\theta d\phi)^2, \quad (3)$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \quad (4)$$

$$\Delta_r = (r^2+a^2)(1-\Lambda r^2 3^{-1}) - 2mr + Q^2 + P^2, \quad (5)$$

$$\Delta_\theta = 1 + \Lambda a^2 3^{-1}\cos^2\theta, \quad (6)$$

$$\Xi = 1 + \Lambda a^2 3^{-1} \quad (7)$$

and m, a, Q and P are constants, m and ma representing mass and angular momentum. Q and P are electric and magnetic charges. The cosmological constant Λ is negative.

One can factorize Δ_r as follows

$$\Delta_r = -\frac{\Lambda}{3}(r-r_0)(r-r_1)(r-r_2)(r-r_3), \quad (8)$$

where at least two roots, say r_0, r_1, are complex conjugates, and we assume r_2 and r_3 are real. If this is the case, then r_2 and r_3 must be positive. One can identify them as the outer black hole and inner black hole horizons, respectively. The roots satisfy the following relations:

$$\sum_i r_i = 0, \tag{9}$$

$$\sum_{i>j} r_i r_j = -\frac{3}{\Lambda} + a^2, \tag{10}$$

$$\sum_{i>j>k} r_i r_j r_k = -\frac{6m}{\Lambda}, \tag{11}$$

$$\prod_i r_i = -\frac{3(a^2 + Q^2 + P^2)}{\Lambda}. \tag{12}$$

The horizon areas are

$$A_i = 4\pi(r_i^2 + a^2)\Xi^{-1}. \tag{13}$$

The surface gravities of the horizons are

$$\kappa_i = \frac{\Lambda \prod_{j\ (j\neq i)}(r_i - r_j)}{6\Xi(r_i^2 + a^2)}. \tag{14}$$

We shall concentrate on the neutral case with $Q = P = 0$ first. The Newman-anti-de Sitter case with nonzero electric or magnetic charge will be discussed later.

The probability of the Kerr-anti-de Sitter black hole pair creation, at the WKB level, is the exponential of the negative of the action of its constrained gravitational instanton.

The constrained instanton is constructed from the complex version of metric (3) by setting $\tau = it$. One can have two cuts at $\tau = \pm\Delta\tau/2$ between the two complex horizons r_0, r_1. Then one makes the f_0-fold cover around the horizon $r = r_0$ and the f_1-fold cover around the horizon $r = r_1$. In order to form a fairly symmetric Euclidean manifold, one can glue these two cuts under the condition

$$f_0\beta_0 + f_1\beta_1 = 0, \tag{15}$$

where we set the imaginary time periods $\beta_0 = 2\pi\kappa_0^{-1}$ and $\beta_1 = 2\pi\kappa_1^{-1}$.

The Lorentzian metric for the black hole pair created is obtained through analytic continuation of the time coordinate from an imaginary to a real value at the equator. The equator is two joint

sections $\tau = consts.$ passing these horizons. It divides the instanton into two halves. We can impose the restriction that the 3-geometry characterized by the parameters m, a, Q and P is given at the equator for the Kerr-Newman-anti-de Sitter family. The parameter f_0 or f_1 is the only degree of freedom left for the pasted manifold, since the field equation holds elsewhere. Thus, in order to check whether we get a stationary action solution for the given horizons, one only needs to see whether the above action is stationary with respect to this parameter. The equator where the quantum transition will occur has topology $S^2 \times S^1$.

The action due to the horizons is

$$I_{i,horizon} = -\frac{\pi(r_i^2 + a^2)(1 - f_i)}{\Xi}. \quad (i = 0, 1) \tag{16}$$

The action due to the volume is

$$I_v = -\frac{f_0 \beta_0 \Lambda}{6\Xi^2}(r_1^3 - r_0^3 + a^2(r_1 - r_0)). \tag{17}$$

If one naively takes the exponential of the negative of half the total action, then the exponential is not identified as the wave function at the creation moment of the black hole pair. The physical reason is that what one can observe is only the angular differentiation, or the relative rotation of the two horizons. This situation is similar to the case of a Kerr black hole pair in the asymptotically flat background. There one can only measure the rotation of the black hole horizon from the spatial infinity. To find the wave function for the given mass and angular momentum one has to make the Fourier transformation [3]

$$\Psi(a, h_{ij}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\delta e^{i\delta J \Xi^{-2}} \Psi(\delta, h_{ij}), \tag{18}$$

where δ is the relative rotation angle for the half time period $f_0\beta_0/2$, which is canonically conjugate to the angular momentum $J = ma$; and the factor Ξ^{-2} is due to the time rescaling. The angle difference δ can be evaluated

$$\delta = \int_0^{f_0\beta_0/2} d\tau(\Omega_0 - \Omega_1), \tag{19}$$

where the angular velocities at the horizons are

$$\Omega_i = \frac{a}{r_i^2 + a^2}. \tag{20}$$

The Fourier transformation is equivalent to adding an extra term into the action for the constrained instanton, and then the total action becomes

$$I = -\pi(r_0^2+a^2)\Xi^{-1} - \pi(r_1^2+a^2)\Xi^{-1} = \pi\left(-\frac{6}{\Lambda} + (r_2^2+a^2)\Xi^{-1} + (r_3^2+a^2)\Xi^{-1}\right). \qquad (21)$$

In the derivation of the second equality in (21) one notices from eqs. (9)(10) that the sum of all horizon areas is equal to $24\pi\Lambda^{-1}$ for all members of the Kerr-Newman-(anti-)de Sitter family. This fact seems coincidental, but it has a deep physical significance.

It is crucial to note that the action is independent of the time identification period $f_0\beta_0$ and therefore, the manifold obtained is qualified as a constrained instanton. Therefore, the relative probability of the Kerr black hole pair creation is

$$P_k \approx \exp -(\pi(r_2^2+a^2)\Xi^{-1} + \pi(r_3^2+a^2)\Xi^{-1}). \qquad (22)$$

It is the exponential of the negative of one quarter of the sum of the outer and inner black hole horizon areas.

Now, let us turn to the charged black hole case. The vector potential can be written as

$$A = \frac{Qr(dt - a\sin^2\theta d\phi) + P\cos\theta(adt - (r^2+a^2)d\phi)}{\rho^2}. \qquad (23)$$

We shall not consider the dyonic case in the following.

One can closely follow the neutral rotating case for calculating the action of the corresponding constrained gravitational instanton. The only difference is to add one more term due to the electromagnetic field to the action of volume. For the magnetic case, it is

$$\frac{f_0\beta_0 P^2}{2\Xi^2}\left(\frac{r_0}{r_0^2+a^2} - \frac{r_1}{r_1^2+a^2}\right) \qquad (24)$$

and for the electric case, it is

$$-\frac{f_0\beta_0 Q^2}{2\Xi^2}\left(\frac{r_0}{r_0^2+a^2} - \frac{r_1}{r_1^2+a^2}\right). \qquad (25)$$

In the magnetic case the vector potential determines the magnetic charge, which is the integral over the S^2 factor. Putting all these contributions together one can find

$$I = -\pi(r_0^2+a^2)\Xi^{-1} - \pi(r_1^2+a^2)\Xi^{-1} \qquad (26)$$

and the relative probability of the pair creation of magnetically charged black holes is written in the same form as eq. (22).

In the electric case, one can only fix the integral

$$\omega = \int A, \tag{27}$$

where the integral is around the S^1 direction. So, what one obtains in this way is $\Psi(\omega, a, h_{ij})$. However, one can get the wave function $\Psi(Q, a, h_{ij})$ for a given electric charge through the Fourier transformation

$$\Psi(Q, a, h_{ij}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{i\omega Q}\Psi(\omega, a, h_{ij}). \tag{28}$$

The Fourier transformation is equivalent to adding one more term to the action

$$\frac{f_0\beta_0 Q^2}{\Xi^2}\left(\frac{r_0}{r_0^2+a^2} - \frac{r_1}{r_1^2+a^2}\right). \tag{29}$$

Then we obtain the same probability formula for the electrically charged rotating black hole pair creation as for the magnetic one. The duality between the magnetic and electric cases is recovered [3][6][7].

The case of the Kerr-Newman black hole family can be thought of as the limit of our case as we let Λ approach 0 from below.

Of course, if one lets the angular momentum be zero, then it is reduced into the Reissner-Nordström-anti-de Sitter black hole case. If one further lets the charge be zero, then it is reduced into the Schwarzschild-anti-de Sitter black hole case.

For the Schwarzschild-anti-de Sitter black hole case, there are only three horizons,

$$r_2 = 2\sqrt{\frac{1}{|\Lambda|}}\sinh\gamma, \tag{30}$$

$$r_1 = \bar{r}_0 = \sqrt{\frac{1}{|\Lambda|}}(-\sinh\gamma - i\sqrt{3}\cosh\gamma), \tag{31}$$

where we set

$$\gamma \equiv \frac{1}{3}\text{arcsinh}(3m|\Lambda|^{1/2}). \tag{32}$$

The horizon $r = r_2$ is the black hole horizon.

The surface gravity κ_i of r_i is [5]

$$\kappa_i = \frac{\Lambda}{6r_i} \prod_{j=0,1,2,(j\neq i)} (r_i - r_j). \tag{33}$$

And the total action is

$$I = -\pi(r_0^2 + r_1^2) = \pi\left(-\frac{6}{\Lambda} + r_2^2\right). \tag{34}$$

Therefore, the relative probability of the pair creation of Schwarzschild-anti-de Sitter black holes, at the WKB level, is the exponential of the negative of one quarter of the black hole horizon area. This contrasts with the case of pair creation of Schwarzschild-de Sitter black holes [3][8]. The relative creation probability for Schwarzschild-de Sitter black holes is the exponential of the total entropy of the universe. One quarter of the black hole horizon area is known to be the entropy in the Schwarzschild-anti-de Sitter universes [9].

One may wonder why we choose horizons r_0 and r_1 to construct the instanton. One can also consider those constructions involving other horizons as the instantons. However, the real part of the action for our choice is always greater than that of the other choices for the given configuration, and the wave function or the probability is determined by the classical orbit with the greatest real part of the action [1]. When we dealt with the Schwarzschild-de Sitter case, the choice of the instanton constructed from the black hole and cosmological horizons had the greatest action accidentally, but we did not appreciate this earlier [3]. This point is important. For example, if, instead we use r_2 and r_3 for constructing the charged or rotating instanton, then the creation probability of a universe without a black hole would be smaller than that with a pair of black holes. This is physically absurd.

In Euclidean quantum gravity the partition function Z is identified with the path integral under the constraints. If the system involves the imposed quantities, namely electric charge Q or (and) angular momentum J, then one has to use the grand partition function Z in grand canonical ensembles for the thermodynamics study [10]. At the WKB level, one has

$$Z = \exp -I, \tag{35}$$

where I is the effective action of a constrained instanton. The effects of the electric charge and angular momentum have been taken into account by the two Fourier transformations.

The entropy S can be obtained

$$S = -\frac{\beta \partial}{\partial \beta} \ln Z + \ln Z, \tag{36}$$

where β is the time identification period.

Thus, the condition that I is independent from β implies that the entropy is the negative of the action. One can use eq. (36) to derive the "entropy" and it is the negative of the action. For compact regular instantons, the fact that the entropy is the negative of the action is shown using different arguments in [11]. For the open creation case, if one naively interprets the horizon areas as the "entropy", then the "entropy" is associated with these two complex horizons. Equivalently, for the chargeless and nonrotating case one can say that the action is identical to one quarter of the black hole horizon area at r_2, or the black hole entropy up to a constant $6\pi\Lambda^{-1}$, as we learn in the Schwarzschild black hole case. For the charged or (and) rotating case, the action is identical to one quarter of the sum of the outer and inner black hole horizon areas up to the same constant.

Our treatment of quantum creation of the Kerr-Newman-anti-de Sitter space family using the constrained instanton can be thought of as a prototype of quantum gravity for an open system, without appealing to the background subtraction approach. The beautiful aspect of our approach is that even in the absence of a general no-boundary proposal for open universes, we treat the creation of the closed and the open universes in the same way.

It can be shown that the probability of the universe creation without a black hole is greater than that with a pair of black holes in the anti-de Sitter background.

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